

D_{00} Propagator in the Coulomb Gauge

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Abstract

The time-time component of the gluon propagator in the Coulomb gauge is believed to provide a long-range confining force. We give the result, including finite parts, for the D_{00} propagator to order g^2 in the Coulomb gauge.

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1. Introduction

The Coulomb gauge has many advantages over the covariant gauges. The propagators are closely related to the polarization states of real spin-1 particles. There are no ghosts states. It is manifestly unitary. Nevertheless, there are problems concerned with the convergence of the energy integrals [1]. The naive Coulomb gauge Feynman rules in non-Abelian gauge theory give rise to ambiguous integrals. At one-loop order and above there are integrals like

$$\int \frac{d^3 P}{(2\pi)^2} \int \frac{dp_0}{(2\pi)} \frac{p_0}{p_0^2 - P^2 + i\eta} \times \frac{1}{(P - K)^2}. \quad (1)$$

There is no regularization procedure for the energy divergence in p_0 . These divergences have been studied in [2],[3] where systematic cancellations have been found. However, ordinary ultra-violet divergences exist along with the above energy divergences. No general proof exists that controls all divergences [4]. This means that a complete treatment of the Coulomb gauge has not yet been given. The Coulomb gauge has been extensively studied in the phase space formalism by D. Zwanziger [5] in the Euclidean space. The ultra-violet divergent parts of the proper two-point functions have been calculated and found to observe the Ward identities. In this paper we give the complete result for the time-time component of the gluon propagator to order g^2 in Minkowski space and explore both limits $k_0 \rightarrow \infty$ and $k_0 \rightarrow 0$ which are believed to give the colour-Coulomb potential.

2. The proper two-point functions

The zero-order propagators used in the Feynman rules in Minkowski space form a 7×7 matrix acting on $A_1, A_2, A_3 ; A_0; E_1, E_2, E_3$:

	A_j	A_0	E_n
A_i	$-T_{ij}/k^2 + \alpha L_{ij}/K^2$	$\alpha k_0 K_i/(K^2)^2$	$-ik_0 T_{in}/k^2$
A_0	$\alpha k_0 K_j/(K^2)^2$	$1/K^2 + \alpha k_0^2/(K^2)^2$	iK_n/K^2
E_m	$-ik_0 T_{mj}/k^2$	iK_m/K^2	$-T_{mn}K^2/k^2$

where

$$T_{ij} \equiv \delta_{ij} - L_{ij}, \quad L_{ij} \equiv K_i K_j / K^2, \quad k^2 = k_0^2 - K^2. \quad (2)$$

The Coulomb gauge propagators are obtained by setting $\alpha = 0$. The field E_i plays the role of the momentum conjugate to A_i . The constants used throughout the paper are

$$\epsilon = 4 - d \quad (3)$$

where d is the dimension of space-time and the coupling parameter is

$$c = \frac{ig^2}{16\pi^2} C_G \delta_{ab}. \quad (4)$$

There are two graphs which contribute to the $A_0 A_0$ function. The graph shown in Fig.1 contributes

$$\begin{aligned}
\Gamma_a^{A_0 A_0} = & c \left\{ \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{-k^2 - i\eta}{\mu^2}\right)^{-\frac{\epsilon}{2}} \times \left(\frac{1}{2}k_0^2 + \frac{5}{6}K^2 + \frac{\epsilon}{12}k^2 + \frac{\epsilon}{6}k_0^2 + \frac{17}{18}\epsilon K^2\right) \right. \\
& - 2^{-\epsilon} \left(\frac{5}{3} + \frac{28}{9}\epsilon\right) \Gamma\left(\frac{\epsilon}{2}\right) K^2 \left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \\
& + \frac{1}{2}k^4 \times D \\
& + \frac{k^4}{2k_0 K} \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{K^2}{(-k^2 - i\eta)} \\
& \left. - k_0^2 \ln \frac{K^2}{(-k^2 - i\eta)} - 2(\ln 2 - 1)k_0^2 \right\} \quad (5)
\end{aligned}$$

The graph in Fig.2 gives

$$\begin{aligned}
\Gamma_b^{A_0 A_0} = & c \left\{ \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{-k^2 - i\eta}{\mu^2}\right)^{-\frac{\epsilon}{2}} \times \left(\frac{1}{3}K^2 - \frac{1}{2}k^2 - \frac{\epsilon}{4}k^2 + \frac{11}{18}\epsilon K^2\right) \right. \\
& - \frac{1}{3} \Gamma\left(\frac{\epsilon}{2}\right) K^2 \left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} - K^2 \left(\frac{10}{9} - \frac{2 \ln 2}{3}\right) \\
& + k^2 k_0^2 \times D \\
& + \frac{k_0 k^2}{K} \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{K^2}{(-k^2 - i\eta)} \\
& \left. - (2k^2 + K^2) \ln \frac{K^2}{(-k^2 - i\eta)} - 2(\ln 2 - 1)(2k^2 + K^2) \right\} \quad (6)
\end{aligned}$$

where the non-rational structure D which appears in the results in the integral form is

$$D = \int_0^1 dx \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \ln(1 - x). \quad (7)$$

In the region $k_0 > K$

$$\begin{aligned}
D = & \frac{1}{k_0 K} \left\{ Li_2\left(\frac{k_0 - K + i\eta}{k_0 + K - i\eta}\right) - Li_2\left(\frac{k_0 + K - i\eta}{k_0 - K + i\eta}\right) \right. \\
& \left. + \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{k^2 + i\eta}{K^2} - i\pi \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \right\}. \quad (8)
\end{aligned}$$

In the region $K > k_0$

$$\begin{aligned}
D = & \frac{1}{k_0 K} \left\{ Li_2\left(\frac{K - k_0 - i\eta}{K + k_0 - i\eta}\right) - Li_2\left(\frac{K + k_0 - i\eta}{K - k_0 - i\eta}\right) \right. \\
& + \ln \frac{K + k_0 - i\eta}{K - k_0 - i\eta} \times \ln\left(\frac{-k^2 - i\eta}{k_0^2}\right) + i\pi \ln \frac{K + k_0 - i\eta}{K - k_0 - i\eta} \left. \vphantom{\frac{1}{k_0 K}} \right\} \\
& - \frac{2}{k_0 K} \left[Li_2\left(-\frac{k_0}{K - i\eta}\right) - Li_2\left(\frac{k_0}{K - i\eta}\right) \right] + \frac{i\pi}{k_0 K} \ln \frac{K^2}{(-k^2 - i\eta)}
\end{aligned} \tag{9}$$

where

$$Li_2(x) = - \int_0^x \frac{\ln(1-z)}{z} dz \tag{10}$$

is the Spence function. The two expressions for D in (8) and (9) are connected as analytic continuations of each other.

3. The time-time propagator

The propagators in the Coulomb gauge are obtained by inverting the 7×7 matrix of the proper two point functions. The complete matrix will be given in a future publication. However, the time-time component of the gluon propagator to order g^2 involves only two more proper functions.

$$\begin{aligned}
D^{A_0 A_0} = & \frac{1}{K^4} [\Gamma^{A_0 A_0} + iK_n \Gamma^{A_0 E_n}] \\
& + \frac{iK_m}{K^4} [\Gamma^{E_m A_0} + iK_n \Gamma^{E_m E_n}]
\end{aligned} \tag{11}$$

To one loop order these proper functions are

$$\Gamma^{A_0 E_i} = c \left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \left\{ \frac{4}{3} - 2^{-\epsilon} \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{2}{3} + \frac{13\epsilon}{9}\right) \right\} \times (2iK_i) \tag{12}$$

and

$$\Gamma^{E_i E_j} = -2c \left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \left\{ 2^{-\epsilon} \left(\frac{2}{3} + \frac{13\epsilon}{9}\right) \Gamma\left(\frac{\epsilon}{2}\right) \delta_{ij} - \frac{4}{3} \frac{K_i K_j}{K^2} \right\} \tag{13}$$

leading to

$$\begin{aligned}
D^{A_0 A_0} = & c(K^2)^{-2} \left\{ \frac{11}{3} \Gamma\left(\frac{\epsilon}{2}\right) K^2 - \frac{5}{3} K^2 \ln \frac{(-k^2 - i\eta)}{\mu^2} - 2K^2 \ln \frac{K^2}{\mu^2} \right. \\
& + \frac{1}{2} k^2 (k^2 + 2k_0^2) \times D \\
& + \frac{k^2}{2k_0 K} (k^2 + 2k_0^2) \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{K^2}{(-k^2 - i\eta)} \\
& \left. - (3k_0^2 - K^2) \ln \frac{K^2}{(-k^2 - i\eta)} - (6k_0^2 + 2K^2) \ln 2 + 6k_0^2 + \frac{31}{9} K^2 \right\}
\end{aligned} \tag{14}$$

The UV divergent part of $D^{A_0 A_0}$ agrees with [5], [6] and [7] showing the antiscreening nature of the QCD vacuum.

4. Discussion

There are two interesting limits of eq.(14). The Zwanziger limit [5] $k_0 \rightarrow \infty$ is

$$\lim_{k_0 \rightarrow \infty} D^{A_0 A_0}(k_0, K) = \frac{c}{K^2} \left\{ \frac{11}{3} \Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3} \ln \frac{K^2}{\mu^2} - i\pi - \frac{28}{3} \ln 2 + \frac{103}{9} - 2 \ln \frac{K}{k_0} \right\} \quad (15)$$

and it is not independent of k_0 . The limit $k_0 \rightarrow 0$ is naturally related to the definition of the quark-antiquark potential. It follows from considering a rectangular Wilson loop with sides of length T in the time direction (where $T \rightarrow \infty$) and L in the space direction. In the Coulomb gauge the main contribution comes from the D_{00} component of the propagator (where $k_0 \rightarrow 0$) attached to the two time-like sides. In this limit ($T \rightarrow \infty$) we expect the contribution to tend to zero for any graph with one end attached to either of the spacelike sides. This leaves graphs connected to the timelike sides at each end. Such graphs involve the D_{00} propagator, which we have calculated, crossing from one side to the other. The contribution of such a loop gives the potential between two very heavy quarks separated by the distance L ,

$$V(L) = -4\pi g_B^2 \int d^{3-\epsilon} K e^{iK \cdot L} D_{00}(0, K) \quad (16)$$

where g_B is the bare coupling constant. The limit $k_0 \rightarrow 0$ of (14) including the zero'th order propagator is

$$\lim_{k_0 \rightarrow 0} D^{A_0 A_0}(k_0, K) = \frac{i}{K^2} \left\{ 1 + c' \left[\frac{11}{3} \Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3} \ln \frac{K^2}{\mu^2} + \frac{31}{9} \right] \right\} \quad (17)$$

where

$$c' = \frac{g^2}{16\pi^2} C_G \delta_{ab}. \quad (18)$$

The bare coupling constant g_B is related to the renormalized coupling constant g_R

$$g_B = \left(1 - \frac{11}{3} \frac{c'}{\epsilon}\right) g_R \mu^{\frac{\epsilon}{2}}. \quad (19)$$

Inserting (17) and (19) into (16) leads to the quark-antiquark potential

$$V(L) = -2\pi^2 g_R^2(\mu) \frac{1}{L} \left\{ 1 + \frac{31}{9} c' + \frac{11}{3} c' \gamma + \frac{11}{3} c' \ln(\mu L)^2 \right\} \quad (20)$$

where γ is the Euler's constant and $g_R(\mu)$ the running coupling constant. If we assume the relation

$$L \times \mu = 1 \quad (21)$$

$g_R(\mu)$ becomes L dependent. We suppose that the exact $g_R(\frac{1}{L})$ tends to zero as $L \rightarrow 0$. Also $g_R(\frac{1}{L}) \rightarrow \infty$ for $L \rightarrow \infty$.

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Figure Captions

Fig.1. The time-time component of the gluon self-energy to order g^2 . The dotted lines represent the instantaneous Coulomb field A_0 . The continuous line is the E_i field conjugate to the transverse field A_i . The propagators inside the loop are the $E_i A_j$ transitions specific to the Coulomb gauge.

Fig.2. Self-energy graph to order g^2 . The dotted line is the A_0 field, the dashed line is the transverse propagator and the solid line is the $E_i E_j$ propagator.

Fig.3 The dashed line is the transverse gluon propagator which attaches to the spacelike sides of length L of the Wilson loop. The timelike sides are of length T . In the limit $T \rightarrow \infty$ the contribution of this loop vanishes.

Fig.4 The dotted line is the zero'th order Coulomb propagator attached to the timelike sides of length T . In the limit $T \rightarrow \infty$ this loop gives the potential between two very heavy quarks which just sit at the positions $(0, 0, 0)$ and $(L, 0, 0)$ for all the time. To zero'th order propagator this just gives the Coulomb potential $\frac{1}{L}$.







